THE YONEDA ALGEBRA OF A $\mathcal{K}_2$ ALGEBRA NEED NOT BE ANOTHER $\mathcal{K}_2$ ALGEBRA

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Abstract. The Yoneda algebra of a Koszul algebra or a D-Koszul algebra is Koszul. $\mathcal{K}_2$ algebras are a natural generalization of Koszul algebras, and one would hope that the Yoneda algebra of a $\mathcal{K}_2$ algebra would be another $\mathcal{K}_2$ algebra. We show that this is not necessarily the case by constructing a monomial $\mathcal{K}_2$ algebra for which the corresponding Yoneda algebra is not $\mathcal{K}_2$.

1. Introduction

Let A be a connected graded algebra over a field K. Correspondences between A and its bigraded Yoneda algebra $E(A) = \bigoplus_{n,m} Ext_A^{n,m}(K,K)$ have been studied in many contexts (e.g. [4], [5], [6] and [10]). In particular there are very interesting classes of algebras where $E(A)$ inherits good properties from A. Perhaps the most famous and intently studied of such classes of algebra is the class of Koszul algebras.

An algebra is Koszul [10] if its Yoneda algebra is generated as an algebra by cohomology degree one elements. Koszul algebras will always have quadratic defining relations and given such an algebra, A, the Yoneda algebra is isomoprhic to the quadratic dual algebra $A^!$. In particular, one has Koszul duality: If A is Koszul then $E(A)$ is Koszul and $E(E(A)) = A$.

The following natural generalization of Koszul was introduced in [2] and also investigated in [7] and [8]. We write $E^n(A)$ for $\bigoplus_p Ext_A^{n,m}(K,K)$.

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Definition 1.1. The graded algebra A is said to be $\mathcal{K}_2$ if $E(A)$ is generated as an algebra by $E^1(A)$ and $E^2(A)$.

Koszul algebras are simply quadratic $\mathcal{K}_2$ algebras. $\mathcal{K}_2$ algebras share many of the nice properties of Koszul algebras, including stability under tensor products, regular normal extensions and graded Ore extensions, (cf. [2]). Every graded complete intersection is a $\mathcal{K}_2$ algebra.

Another important class of algebras is the class of D-Koszul algebras introduced by Berger in [1]. This is the class defined by: $Ext_A^{n,m}(K,K) = 0$ unless $m = \delta(n)$, where $\delta(2n) = nD$ and $\delta(2n+1) = nD+1$. These algebras arise naturally in certain contexts and all such D-Koszul algebras are easily seen to be $\mathcal{K}_2$. A remarkable theorem in [3] states that if A is D-Koszul algebra, then $E(A)$ is a $\mathcal{K}_2$ algebra, and furthermore, it is possible to regrade $E(A)$ in such a way that $E(A)$ becomes a Koszul algebra. In particular one gets a "delayed" duality: $E(E(A)) = E(A)^!$ and $E(E(E(A))) = E(A)$.

Based on the above theorem of [3], Koszul duality, and calculations of many other $\mathcal{K}_2$-examples, it seems reasonable to hope that the Yoneda algebra of any $\mathcal{K}_2$ algebra would also be $\mathcal{K}_2$, perhaps even Koszul. Unfortunately, this is not always the case, and the purpose of this article is to exhibit an example of a $\mathcal{K}_2$ algebra for which the corresponding Yoneda algebra is not Koszul nor even $\mathcal{K}_2$. Our example has 13 generators and 9 monomial defining relations. We believe that such a monomial algebra cannot be constructed with fewer generators and relations.

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2. The algebras $A, E(A)$ and $E(E(A))$

Let K be a field. Let $\{m,n,p,q,r,s,t,u,v,w,x,y,z\}$ be a basis for a vector space V. We define A to be the K-algebra $T(V)/I$ where I is the ideal generated by this list of monomial tensors:

$$R = \{mn^2p,\ n^2pqr,\ npqrs,\ pqrst,\ stu,\ tuvwx,\ uvwxy,\ vwxy^2,\ xy^2z\}.$$

Theorem 2.1. *The algebra A is $\mathcal{K}_2$, but the algebra $E(A)$ is not $\mathcal{K}_2$.*

Proof. We use the algorithm given in section 5 of [2] to prove that A is $\mathcal{K}_2$. From the set R one can calculate that $S_1 = \{m,n,p,q,r,s,t,u,v,w,x,y,z\}$, $S_2 = \{mn^2,\ n^2pq,\ npqr,\ pqrs,\ st,\ tuvw,\ uvwx,\ vwxy,\ xy^2\}$, $S_3 = \{pqr,\ vw\}$, $S_4 = \{n^2\}$ and $S_5 = \emptyset$. One easily verifies that for every $b \in S$ with minimal left annihilator a we have either $\deg(a) = 1$ or $ab \in R$, and hence A is $\mathcal{K}_2$.

Let $B = E(A)$. In what follows we consider only the cohomology grading on B. Following section 5 of [2] we can construct a minimal projective resolution $P^\bullet$ of ${}_AK$ and see that the Hilbert Series of the algebra B is $1 + 13t^2 + 9t^2 + 8t^3 + 4t^4 + 3t^5 + t^6$.

It is possible (although laborious) to describe B in terms of generators and relations and then construct a minimal resolution of ${}_BK$ and apply Theorem 4.4 of [2] to show that B is not $\mathcal{K}_2$. However B's failure to be $\mathcal{K}_2$ is apparent already in $Ext_B^3(K,K)$ and consequently there is a more efficient way for us to illustrate this.

Let $\bar{m}$ and $\bar{z}$ denote the basis elements in B_1 dual to m and z in A_1. The vector space B_2 has a basis dual to the elements of the list of relations R. We will use α, β and γ to denote the dual basis elements corresponding to the monomials n^2pqr, stu and $vwxy^2$. From the maps in the resolution $P^\bullet$ one can see that $\bar{m}\alpha$, $\gamma\bar{z}$ and $\beta\gamma$ are nonzero in B, while $\bar{m}\alpha\beta$ and $\beta\gamma\bar{z}$ are each zero.

Recall that $Tor^B(K,K)$ can be calculated using the bar-complex [9] where $\mathcal{B}ar_i(K,B,K) = K\otimes_B B\otimes B_+\otimes\cdots\otimes B_+\otimes K = B_+^{\otimes i}$. Let $\zeta = \bar{m}\alpha\otimes\beta\gamma\otimes\bar{z} \in B_+^{\otimes 3}$. The differential on the bar-complex gives us $\partial(\zeta) = \bar{m}\alpha\beta\gamma \otimes \bar{z} - \bar{m}\alpha \otimes \beta\gamma\bar{z} = 0$. ζ is not in the image of $B_+^{\otimes 4}$ because $\partial(\bar{m}\otimes\alpha\otimes\beta\gamma\otimes\bar{z}) = \bar{m}\alpha\otimes\beta\gamma\otimes\bar{z} - \bar{m}\otimes\alpha\beta\gamma\otimes\bar{z}$ while $\partial(\bar{m}\alpha\otimes\beta\otimes\gamma\otimes\bar{z}) = -\bar{m}\alpha\otimes\beta\gamma\otimes\bar{z} + \bar{m}\alpha\otimes\beta\otimes\gamma\bar{z}$. Thus ζ represents a non-zero homology class in Tor_3^B.

In contrast the element $\bar{m}\alpha\otimes\beta\gamma = \partial(-\bar{m}\alpha\otimes\beta\otimes\gamma)$ represents zero in Tor_2^B and $\bar{m}\alpha = \partial(\bar{m}\otimes\alpha)$ represents zero in Tor_1^B. Therefore under the co-multiplication map

$$\Delta : Tor_3^B(K,K) \to Tor_2^B(K,K)\otimes Tor_1^B(K,K) \oplus Tor_1^B(K,K)\otimes Tor_2^B(K,K)$$

we have $\Delta(\zeta) = 0$. This failure of Δ to be injective is equivalent to the multiplication map

$$E^2(B)\otimes E^1(B) \oplus E^1(B)\otimes E^2(B) \to E^3(B)$$

not being surjective. Hence $E(B)$ is not generated by $E^1(B)$ and $E^2(B)$, and so B is not a $\mathcal{K}_2$ algebra. □

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